

Epicyclic frequencies for rotating strange quark stars: the importance of stellar oblateness

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Kilohertz QPOs can be used as a probe of the inner regions of accretion disks in compact stars and hence also of the properties of the central object. Most models of kHz QPOs involve epicyclic frequencies to explain their origin. We compute the epicyclic frequencies of nearly circular orbits around rotating strange quark stars. The MIT bag model is used to model the equation of state of quark matter and the uniformly rotating stellar configurations are computed in full general relativity. The vertical epicyclic frequency and the related nodal precession rate of inclined orbits are very sensitive to the oblateness of the rotating star. For slowly rotating stellar models of moderate and high mass strange stars, the sense of the nodal precession changes at a certain rotation rate. At lower stellar rotation rates the orbital nodal precession is prograde, as it is in the Kerr metric, while at higher rotation rates the precession is retrograde, as it is for Maclaurin spheroids. Thus, qualitatively, the orbits around rapidly rotating strange quark stars are affected more strongly by the effects of stellar oblateness than by the effects of general relativity. We show that epicyclic and orbital frequencies calculated numerically for small mass strange stars are in very good agreement with analytical formulae for Maclaurin spheroids.

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I. INTRODUCTION

Strange quark stars (SQS) are considered as a possible alternative to neutron stars as compact objects (see, e.g., [1] for a review). The possibility of the existence of quark matter was first recognized in the early seventies. Bodmer [2] remarked that matter consisting of deconfined up, down and strange quarks could be the absolute ground state of matter at zero pressure and temperature. If this is true, then macroscopic stellar-mass objects made of such matter, i.e., quark stars (also called “strange stars”) could in principle exist [3]. Typically, strange stars [4, 5] are modeled with an equation of state (EOS) based on the phenomenological MIT-bag model of quark matter in which quark confinement is described by an energy term proportional to the volume [6].

It was shown [7] that a strange star described by the standard MIT bag model can be accelerated to high rotation rates in low-mass X-ray binaries (LMXBs) taking into account both a reasonable value of mass of the strange quark, and secular instabilities, such as the viscosity-driven instability and the r -mode instability. Therefore, strange stars in LMXBs could rotate rapidly (with spin frequency > 400 Hz). This provides the astrophysical motivation for computing models of rapidly rotating strange stars.

General relativity (GR) predicts the existence of the marginally stable orbit, within which no stable circular motion is possible [8], and all models of accretion disks

around black holes take this into account; e.g. [9, 10]. In the case of neutron stars, the marginally stable orbit may be separated from the stellar surface by a gap, whose size depends on the equation of state of dense matter and the spin of the neutron star, as well as on its mass [11, 12]. Whether or not the accreting fluid attains that orbit depends also on the value of the stellar magnetic field [13]. Similar considerations apply to quark stars [14].

The marginally stable orbit is often called the innermost stable circular orbit (ISCO). While in the Kerr geometry [15] this can lead to no misunderstandings, in general an ISCO cannot be identified with the marginally stable orbit. In some metrics a marginally stable orbit may be the outermost (in a certain radius range) stable circular orbit [16, 17], while in the Newtonian gravity of a $1/r$ potential all circular orbits are stable, and the innermost one is simply the one grazing the surface of the spherical gravitating body. In this paper we will mostly avoid using the term ISCO in the context of quark stars, where the term ISCO is unambiguous only when there is a gap between the marginally stable orbit and the stellar surface.

Whenever the marginally stable orbit is present around a neutron star or a quark star, its frequency is an upper bound on the frequency of stable orbital motion of a test particle. In addition to the orbital frequencies, epicyclic frequencies are of great interest in the discussion of accretion disks in GR. Indeed, the Rayleigh criterion for stability of circular motion is that $\nu_r^2 > 0$. Thus, the radial epicyclic frequency, ν_r , goes to zero at the marginally stable orbit, and therefore must have a

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maximum at a somewhat larger radius.¹ The presence of a maximum in ν_r allows mode trapping of g -modes of disk oscillations, whose eigenfrequency is somewhat lower than the maximum value of ν_r [18, 19], while the vertical epicyclic frequency is related to a generalization of the Lense-Thirring precession, the so called c -mode, whose eigenfrequency is approximately equal to the difference between the orbital and the vertical epicyclic frequencies [20]. Such modes may have been detected in LMXBs as the celebrated kHz QPOs (see e.g., [21] for a review of QPOs).

In Newtonian gravity, all circular orbits around spherically symmetric objects are stable. However, the marginally stable orbit may be present around *rapidly rotating* Newtonian stars [22, 23]. Indeed, for rapidly rotating Maclaurin spheroids, the ISCO is well outside the surface of this figure of equilibrium [24].

In this paper we report on numerical calculations in general relativity of epicyclic frequencies for rotating strange stars with astrophysically relevant masses of $1.4M_\odot$ and $1.96M_\odot$. We use an up-to-date version of the RNS code [25]. The motivation for the study is explained in § II, and the implications of our findings are sketched in § VI.

II. AIMS OF THE STUDY

The primary purpose of this work is to understand the influence of rotation-induced oblateness on the orbital and epicyclic frequencies in rapidly rotating relativistic stars. In particular, we would like to clarify the origin of the departures of these frequencies from their Kerr values. Astrophysically this is interesting, for instance in the context of similarities and differences between the kHz QPOs in LMXB neutron stars and their counterparts in black hole X-ray sources. Previously, in the context of neutron star QPOs, it had been expected that frame-dragging effects dominate those of oblateness [26, 27], as they do in the Hartle-Thorne metric [28].

At the same time, we would like to validate the RNS code in the exacting regime of the low-mass Newtonian limit. In fact we are in the fortunate position of having exact analytic formulae [29] against which numerical values of the frequencies can be compared. Previously, the only test available for the epicyclic frequency module was given by the position of the ISCO, for which the RNS numerical value could be compared with the known formulae for the Schwarzschild and Hartle-Thorne metrics, or with the numerical values obtained in the LORENE code [30] (however, LORENE does not as yet allow computation of the epicyclic frequencies).

For these two reasons we have chosen to study quark

star models. Quark matter, being self-bound (if it exists), admits stellar models of arbitrarily low mass, while neutron stars being bound by gravity are intrinsically relativistic objects (in the sense of GR) and are unstable at low masses. We have performed calculations for extremely low-mass models of gravitational mass $M = 0.001M_\odot$ and $0.01M_\odot$ to check the accuracy of our relativistic code in the Newtonian limit. Our numerical calculations of the orbital and epicyclic frequencies around low mass stars agree very well with the analytical calculations for Maclaurin spheroids [24, 29], thus validating the code in the low-mass limit as well as its epicyclic frequency module in general.

Further, quark stars have higher density than neutron stars and (apart from a possible thin baryonic crust) are devoid of low density envelopes. Hence, the effects of stellar oblateness related to rotation are more pronounced than in neutron stars. This allows a fairly clear identification of the oblateness contribution, as we will see in the results of our calculations of models with astrophysically relevant masses.

Of course, having at hand results for rapidly rotating quark-star models, we can try to speculate on the implications for observed phenomena, such as QPOs, and we do so briefly in § VI. It has long been recognized that a secure identification of quark stars would have profound implications for our view of the stability of hadronic matter [2, 3], and it is in this spirit that predicted observational signatures of quark stars have been studied; e.g., [31, 32].

III. EQUATION OF STATE OF STRANGE STARS

We perform all numerical calculations of quark stars in the framework of the MIT bag model, within which the quark matter is composed of massless up and down quarks, massive strange quarks, and electrons. There are three physical quantities describing the model: the mass of the strange quarks, m_s , the bag constant, B , and the strength of the QCD coupling constant α . We use a simple MIT bag model where $m_s = 0$, $\alpha = 0$ and $B = 60 \text{ MeV/fm}^3$, with the equation of state given by

$$P = a(\rho - \rho_0)c^2, \quad (1)$$

where P is the pressure, ρ the mass-energy density, and c is the speed of light. Both ρ_0 and a are functions of the physical constants B , m_s and α . In our case $a = 1/3$ and $\rho_0 = 4.2785 \times 10^{14} \text{ g/cm}^3$. Essentially, Eq. (1) corresponds to self-bound matter with density ρ_0 at zero pressure and with a fixed sound velocity ($\sqrt{a}c$) at all pressures. For a fixed value of a , all stellar parameters are subject to scaling relations with appropriate powers of ρ_0 . For instance, $f \propto \rho_0^{1/2}$, where f denotes either of the rotational or orbital frequencies, and $M, R \propto \rho_0^{-1/2}$ (e.g., [3, 33]).

¹ In Schwarzschild geometry the ISCO is at $r = 6M(G/c^2)$, while the maximum of ν_r is attained at $r = 8M(G/c^2)$.

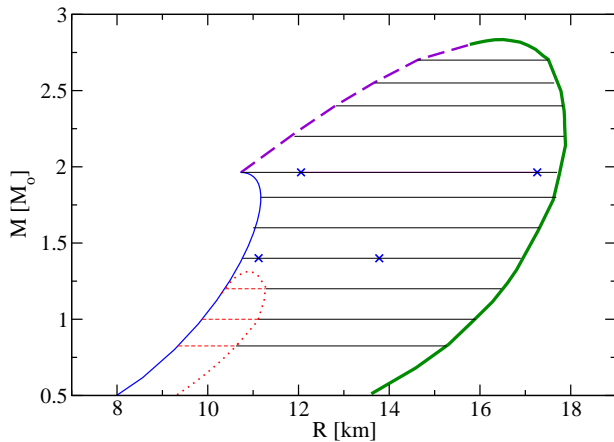


FIG. 1: Gravitational mass versus equatorial radius for static (thin blue line) and uniformly rotating strange quark stars described by the MIT bag model. Horizontal lines correspond to sequences with a constant gravitational mass. The radius increases with rotation rate until the mass-shedding limit is reached (thick green line). Both normal and supramassive sequences are shown. Thick solid violet long-dashed line corresponds to configurations marginally stable to axisymmetric perturbations. The crosses represent models considered in detail in the paper: two with gravitational mass $1.4M_\odot$, rotating at frequencies 600 Hz and 1165 Hz (from left to right), and two with $1.964M_\odot$ at 910 Hz and 1252 Hz.

IV. ROTATING STRANGE QUARK STARS

A. Numerical models

We have calculated axisymmetric models of rotating strange quark stars and their exterior metrics using a highly accurate relativistic code, RNS ([25], see [34] for a description). In this code the equilibrium models are obtained following the KEH method [35], with extensions by [12], in which the field equations are converted to integral equations using appropriate Green's functions. The code was applied to perform calculations of epicyclic frequencies for neutron stars in [36], and the extended RNS code was tested against selected results published in [37].

We have computed the metric outside uniformly rotating quark stars of masses and rotation rates that may be typical of compact stars which have been spun up through accretion in LMXBs. In this paper we will consider in detail the orbital and epicyclic frequencies of test particles for four models of quark stars with masses and rotation rates likely to occur in LMXBs, if there are LMXBs containing quark stars. These are two models of stars with gravitational mass $M = 1.4M_\odot$, rotating at frequencies $\nu_* = 600$ Hz and at $\nu_* = 1165$ Hz, and two

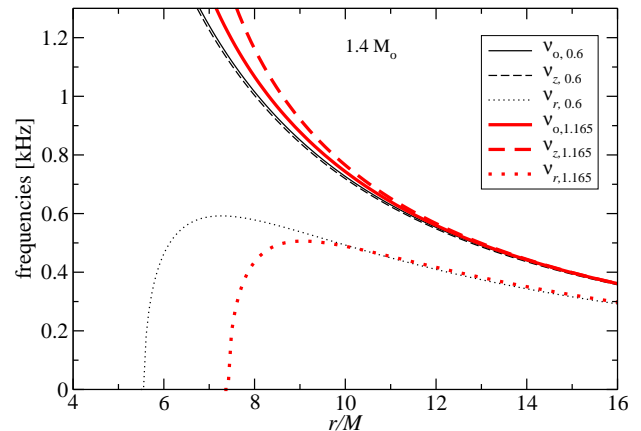


FIG. 2: Orbital and epicyclic frequencies versus radius (scaled with gravitational stellar mass M) for numerical models of an $M = 1.4M_\odot$ uniformly rotating strange quark star rotating at a fixed frequency, 600 Hz (thin black lines) and 1165 Hz (thick red lines). From top to bottom: vertical epicyclic frequency (dashed lines), orbital frequency (solid lines), radial epicyclic frequency (dotted lines).

models with $M = 1.964M_\odot$ rotating at $\nu_* = 910$ Hz and $\nu_* = 1252$ Hz. These models are indicated by crosses in Fig. 1 (from left to right and bottom to top).

The general relationship of gravitational mass to the equatorial radius for uniformly rotating SQS described by the MIT bag model is shown in Fig. 1. Each horizontal (thin solid or short-dashed) line corresponds to a sequence with constant gravitational mass. For each sequence the largest radius is attained for the equatorial mass-shedding models (thick green solid line), while the smallest radius model is either the static model (left-most, thin, blue solid curve) or a rotating configuration marginally stable to the axisymmetric perturbations (thick violet long-dashed line). Models with higher baryon mass than the baryon mass of static neutron star with maximum gravitational mass are called supramassive. The angular momentum increases along each sequence from $J = 0$ for static configurations, or $J_{\min} > 0$ on the axisymmetric stability limit, to J_{\max} for configurations at the mass-shedding limit. Mass shedding occurs when the angular velocity of the star reaches the angular velocity of a particle in a circular Keplerian orbit at the equator (for SQS this is an unstable orbit).

Sequences shown with short-dashed red lines correspond to rotating models for which the marginally stable orbit does not exist. The limiting configurations for which the radius of the marginally stable orbit coincides with the stellar equatorial radius, i.e., for which $r_{\text{ms}} = R_{\text{eq}}$, are shown by the thin red dotted line in

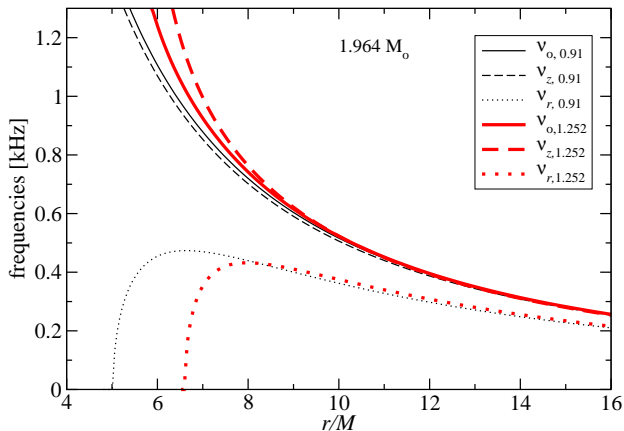


FIG. 3: Same as Fig. 2, but for $1.964 M_\odot$ at 910 Hz (thin black lines) and 1252 Hz (thick red lines).

Fig. 1. This line separates the models with and without a marginally stable orbit. It is evident that the marginally stable orbit exists around SQS for a broad range of masses and rotation rates and is absent only in slowly rotating, low-mass strange quark stars.

B. Orbital and epicyclic frequencies

If one recalls that the radius of the marginally stable orbit decreases with increasing black hole spin in the Kerr metric, and that the stellar radius increases with increasing stellar angular momentum, it may seem counter-intuitive that the gap between the marginally stable orbit and the stellar surface disappears for low stellar rotation rates, while it is present for rapidly rotating models. Of course, the space-time of a rotating star is not described by the Kerr metric, and the complex behavior of the marginally stable orbit, and more generally of the epicyclic frequencies for rapidly rotating compact stars, is the focus of this paper.

We denote the orbital frequency by ν_o , the vertical epicyclic frequency by ν_z and the radial epicyclic frequency by ν_r . When comparing them with the angular frequencies quoted in the literature, one should bear in mind that the latter quantities are larger by a factor of 2π , e.g., the angular frequency of the radial epicycle is $\kappa = 2\pi\nu_r$.

The orbital and epicyclic frequencies of test particles in such a metric is exhibited for two $M = 1.4 M_\odot$ models in Fig. 2. The leftmost black curves correspond to a stellar rotational frequency of 600 Hz. This is a modest rotation rate, so the star may reasonably be described by the Hartle-Thorne [28] metric, which coincides through the

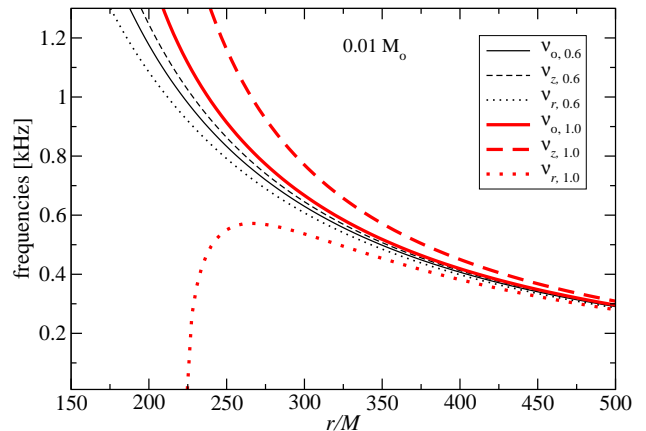


FIG. 4: Same as Fig. 2, but for $0.01 M_\odot$ at 600 Hz (thin black lines) and 1165 Hz (thick red lines).

first order in stellar spin with the Kerr metric (see also [11]). As expected, a marginally stable orbit is present slightly inwards of $r = 6M$ (the location where ν_r , the black dotted curve in Fig. 2, goes through zero), and the vertical epicyclic frequency is slightly *lower* than the orbital frequency.² These results are consistent with those for a black hole with a small value of the dimensionless Kerr parameter, $0 < a_* < 1$ (e.g., [20]).

The same quantities are shown in thick red lines for a star of the same mass, but spinning nearly twice as fast, at 1165 Hz. Doubling the spin in the Kerr metric would lead to a smaller ISCO radius, a higher maximum value of the radial epicyclic frequency and a larger splitting between the vertical epicyclic frequency and the orbital one, with $\nu_z < \nu_o$. However, as is apparent from Fig. 2, doubling the rotational rate of the quark star leads to an increase of the radius of the marginally stable orbit and a lowering of the maximum value of ν_r . The splitting between the vertical epicyclic and the orbital frequencies does indeed increase, but the *relative ordering* changes (for the radius range shown in the figure), from $\nu_z < \nu_o$ at the 600 Hz rotation rate, to $\nu_z > \nu_o$ at the 1165 Hz rate.

The same effects are exhibited by larger mass quark stars. Fig. 3 shows the frequencies for $1.964 M_\odot$ models rotating at 910 Hz (black curves) and at 1252 Hz (red thick curves). Again, at the lower rotation rate the frequencies are qualitatively similar to those of the Kerr metric, with the radius of the marginally stable orbit

² We suppress factors of G/c^2 in the text when discussing ratios such as r/M , and $J/M^2 \equiv a_*$.

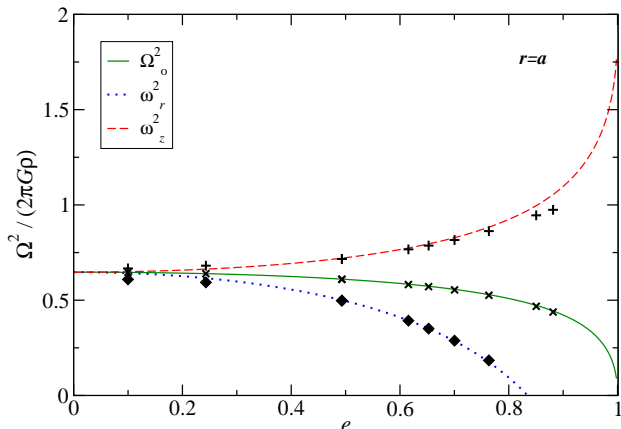


FIG. 5: Orbital and epicyclic frequencies (squared, in units of $2\pi G\rho$) at the stellar equator ($r = R_{\text{eq}} \equiv a$) as a function of ellipticity of the Maclaurin spheroids (solid lines). Also shown (various symbols) numerical models of uniformly rotating strange quark stars of mass $M = 0.001M_{\odot}$ calculated by us with the general relativistic code RNS. From top to bottom: vertical epicyclic frequency (red dashed line and + crosses), orbital frequency (green solid line and x crosses), radial epicyclic frequency (blue dotted line and diamonds).

$r_{\text{ms}} < 6M$, and $\nu_z < \nu_o$. However, at the higher rotation rate, $r_{\text{ms}} > 6M$ and $\nu_z > \nu_o$.

Clearly, these departures from a Kerr-metric behavior must be related to the presence of a rotationally distorted (“flattened”) figure of equilibrium of the fluid, i.e., a non-spherical mass distribution with its higher gravitational multipoles. To test this claim, we have computed the metric of a very low mass quark star, of $M = 0.01M_{\odot}$. Such low masses are possible for bodies composed of quark fluid, for unlike neutron star matter which is unstable in the absence of strong gravity, quark matter is self-bound (according to the Bodmer–Witten strange matter hypothesis [2, 3]).

Fig. 4 exhibits the frequencies of test particle orbits for two $0.01M_{\odot}$ models. For stars of such low mass, the orbits essentially coincide with those in Newtonian gravity. For slowly rotating fluid configurations, departures from spherical symmetry should remove the degeneracy of the epicyclic frequencies and the orbital one that is present in a $1/r$ potential, but all circular orbits should be stable, i.e., the marginally stable orbit should be absent. This is indeed the case for the 600 Hz model (shown in black), but notably not true for a rapidly rotating model. At 1000 Hz the radial epicyclic frequency reaches a maximum at about $r \approx 270M$ and goes to zero at $r_{\text{ms}} = 225M$. Note that these radii are much larger than the ones in Figs. 2 and 3, which are comparable to the Schwarzschild values. Here, instead of being a low mul-

tipole of the gravitational radius M , the marginally stable orbit radius is comparable to the equatorial radius of the star $r_{\text{ms}} \approx R_{\text{eq}}$. Clearly, at 1000 Hz the rotational distortion of this low-mass star is sufficiently high for the Newtonian marginally stable orbit to appear.³ We note that regardless of the rotation rate, the ordering of the orbital and the vertical epicyclic frequencies is $\nu_r < \nu_o < \nu_z$ for this low mass model. As recently shown in [29], this is the ordering of frequencies in circular orbits in the Newtonian gravitational field of the classic Maclaurin spheroids. Therefore, we now turn to a discussion of the Newtonian limit of rotating quark star models.

V. NEWTONIAN LIMIT

To discuss the Newtonian limit of our numerical results we compare them with the known results for the classic figure of equilibrium of a uniformly rotating fluid body, with uniform density ρ . The analytic formulae for orbital and epicyclic frequencies for test particles in motion around Maclaurin spheroids can be found in ref. [29]. We plot the squares of these frequencies as a function of the ellipticity of the spheroid in Figs. 5 and 6, for three different radii of the orbits, $r = a$, $r = 1.3a$, and $r = 2a$ (a being the equatorial radius of the spheroid). All frequencies scale with the square root of the (uniform) density of the spheroid. Overplotted with crosses and diamonds are the corresponding values computed with our numerical code for a sequence of $M = 0.001M_{\odot}$ quark star models. The ellipticity, e , is defined by $e^2 = 1 - R_p^2/R_{\text{eq}}^2$, where R_p and R_{eq} are the polar and equatorial radii of the spheroid, respectively. We define *oblateness* as $1 - R_p/R_{\text{eq}}$. The overall agreement of our numerical results with the analytic formulae is excellent, the only significant discrepancies occurring for orbits grazing the equator (i.e., at $r = a$) at the highest ellipticities, and even then only for the vertical epicyclic frequency⁴.

We continue the comparison in Fig. 7, now plotting the scaled frequencies squared as a function of the radius (in units of the equatorial radius) at fixed ellipticities of $e = 0.6$ and $e = 0.954$. Again, there is excellent agreement between the numerical results obtained with the RNS code for relativistic quark star models and the analytic formulae for Maclaurin spheroids. Note the presence of the marginally stable orbit ($\nu_r^2 = 0$) outside the stellar surface for $e = 0.954$, but not for $e = 0.6$. This is in

³ In Newtonian gravity, the appearance of a marginally stable orbit requires a sizable octupole moment [22].

⁴ Since we are using relativistic SQS models, such a level of agreement with uniform density Maclaurin spheroids is still remarkable and the slight discrepancy in the vertical epicyclic frequency at the surface of the star indicates that this particular frequency is sensitive to the precise multipolar structure of the gravitational field.

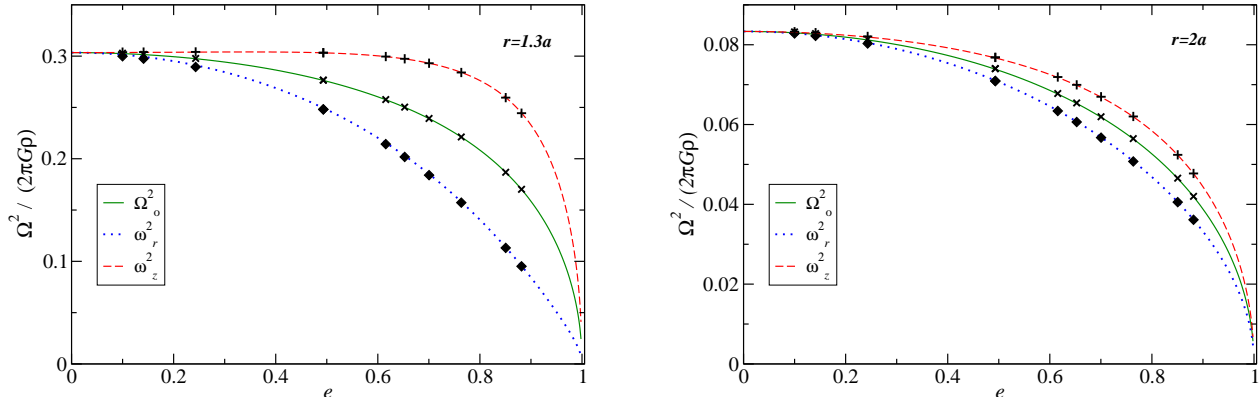


FIG. 6: Same as Fig. 5 but at $r = 1.3a$ (left panel) and at $r = 2a$ (right panel).

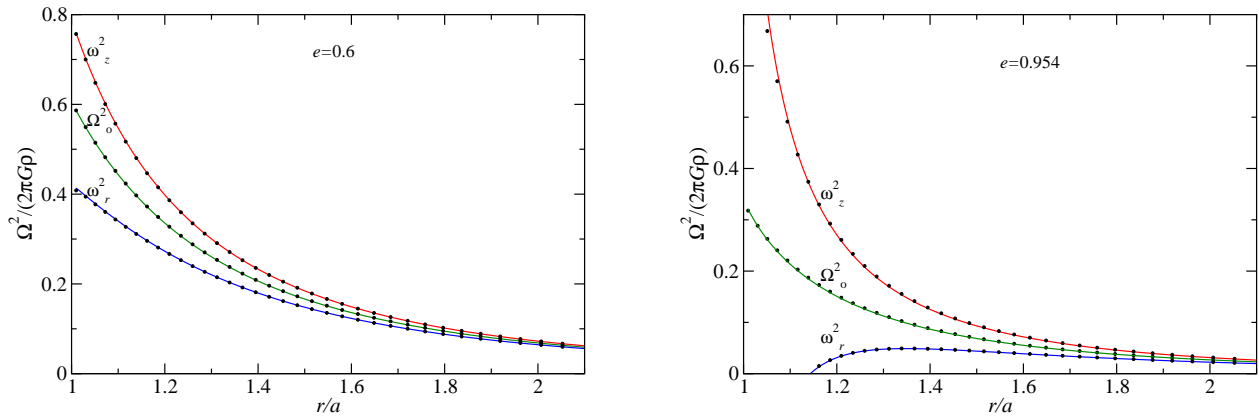


FIG. 7: Orbital and epicyclic frequencies (squared, in units of $2\pi G\rho$) versus radius (scaled with equatorial radius a) for Maclaurin spheroids (solid lines) and numerical models of relativistic, uniformly rotating strange quark stars of mass $M = 0.001M_\odot$, described by the MIT bag model (dots). For scaling purposes, the value of ρ for SQS is here assumed to be equal to the central density of the configuration. Left panel: stars with ellipticity $e = 0.6$ (oblateness 0.2). Right panel: $e = 0.954$ (oblateness 0.7). From top to bottom: vertical epicyclic frequency (red), orbital frequency (green), radial epicyclic frequency (blue).

agreement with Fig. 5 where, clearly, $\nu_r^2 < 0$ for $e = 0.954$ at $r = a$, and $\nu_r^2 > 0$ for $e = 0.6$ at $r = a$, implying instability of circular orbits at the equator in the former case and stability of orbits at the equator in the latter case.

As remarked in [29], the overall relationship of the epicyclic and orbital frequencies is reminiscent of retrograde orbits in the Kerr geometry. However, in the Newtonian limit represented in Fig. 7, there is no distinction between prograde and retrograde orbits.

VI. ASTROPHYSICAL IMPLICATIONS

We have found that rotationally induced oblateness strongly affects the epicyclic frequencies in relativistic stars. It has already been noted in the literature that the position of the marginally stable orbit is pushed out by rapid stellar rotation [32]. We now note that this is related to a general decrease of the radial epicyclic frequency. In contrast, the vertical epicyclic frequency increases with the oblateness of the star, thus the difference between the two epicyclic frequencies increases with the stellar rotation rate. These effects may have interest-

ing astrophysical implications, particularly if it turns out that similar qualitative effects obtain for stars modelled with the conventional neutron star matter equations of state (this is a subject of an ongoing investigation).

High frequency QPOs (HFQPOs) are variations in the light curves of accreting neutron stars and black holes [21]. In LMXBs, because of low photon counts (i.e., small area of currently existing detectors) they are currently observed only in the Fourier transform of the light curves, as peaks of finite width in the power density spectra⁵. Their quality factor varies from a few in the black hole case, to $Q > 100$ in neutron stars [39]. Very often the HFQPOs occur in pairs, in neutron star sources the two frequencies correlate in a reproducible fashion with other properties of the source (such as luminosity).

Nearly all models of HFQPOs in neutron stars and black holes involve some combination of the epicyclic frequencies and the orbital frequency. Therefore, a marked change of the epicyclic frequencies related to the stellar oblateness will affect the predictions of the models, as well as the differences between HFQPOs in the neutron/quark star and black hole systems (if the models are correct).

In the relativistic precession model (RPM) [40], the upper kHz QPO is taken to be the orbital frequency, while the lower one the difference between the orbital frequency and the radial epicyclic frequency. Applying this model to our results for quark stars, one would expect that the lower kHz QPO should increase in frequency with the rotation rate of the star. This in itself would not be a dramatic effect. However, in general, if at least one frequency involves the orbital frequency, one would expect a cut-off at the marginally stable orbit [13], and since the stellar mass is usually inferred in such a case from the highest observed frequency, which is taken to be the orbital frequency in the marginally stable orbit [13, 26, 41], M would have been overestimated if in fact the marginally stable orbit is pushed out by rapid stellar rotation, as we find for the SQS models.

In diskoseismology [19, 42], the lower kHz QPO is identified with the g -mode [18] trapped near the maximum of the radial epicyclic frequency, while the higher kHz QPO with a c -mode [20]. For rapidly rotating quark stars, the frequency of the lower kHz QPO would then be diminished in this model (in contrast to the RPM). Perhaps the most prominent effects would be on the upper kHz QPO if it corresponds to a *trapped* c -mode. It has been demonstrated [20] that no trapped c -modes exist if $\nu_z - \nu_o > 0$ is a decreasing function with r , as is indeed the case close to the rotating quark star (Fig. 8). In this model it seems likely that the higher kHz QPO would be absent in the emissions of the inner accre-

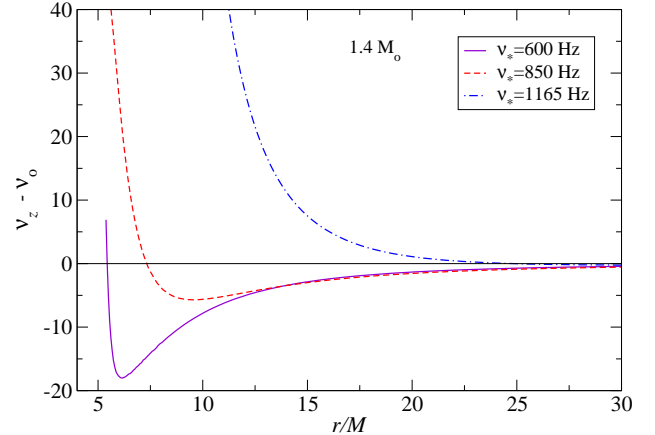


FIG. 8: The difference between vertical epicyclic frequency and orbital frequency for a $1.4M_\odot$ rigidly rotating strange quark star rotating with frequency 1165 Hz (blue dot-dashed line), 850 Hz (red dashed line) and 600 Hz (violet solid line).

tion disk of rapidly rotating quark stars. If the c -modes are absent down to the minimum of $\nu_z - \nu_o$, Figs. 8, 2 suggest that already for a stellar rotation rate lower than 900 Hz, the upper kHz QPO could be lower by about 300 Hz than for slowly rotating stars (because the c -mode would truncate at $r \approx 9M$ instead of $r \approx 6M$). However, the status of c -modes in the case when $\nu_o - \nu_z > 0$ is an increasing function of r is unclear at present. The carefully investigated case [20] is that of the Kerr metric, where $\nu_z - \nu_o < 0$ and $\nu_o - \nu_z$ is a decreasing function of r for an accretion disk co-rotating with the black hole. More definitive conclusions must await a consensus on the origin of kHz QPOs.

VII. DISCUSSION

In general relativity, the properties of the marginally stable orbit are familiar from work in the Kerr metric [15]. For prograde orbits the radius of the ISCO, in units of the gravitational mass, is a monotonically decreasing function of the black hole spin, a_* . For retrograde orbits, it is a monotonically increasing function of a_* . In the Schwarzschild metric the vertical epicyclic frequency is equal to the orbital frequency. In the Kerr metric, the vertical epicyclic frequency is lower than the orbital frequency for prograde orbits and larger for retrograde orbits (e.g., [20]). The lower the value of the ISCO radius, the higher the maximum value of the epicyclic frequency.

We found that models of rapidly rotating SQS of astrophysically relevant masses ($M_\odot < M < 2M_\odot$) have epicyclic frequencies whose behavior differs qualitatively from that of prograde orbits in the Kerr geometry. The

⁵ HFQPOs seem to have their counterparts in two supermassive black holes, where they can be observed directly in the light curve. These are Sgr A* at the centre of the Galaxy, with its 17-minute QPO, and NGC 5408 X-1 [38].

marginally stable orbit is pushed away from the star to values $r/M > 6$, while near the star the vertical epicyclic frequency is larger than the orbital frequency (Figs. 2, 3).

Qualitatively similar statements hold for rotating neutron stars, although the effect of rotation is less pronounced [36]. Stergioulas, Kluźniak, Bulik [32] computed the location of the ISCO for rapidly rotating massive quark stars and noted that rotation-induced oblateness pushes the ISCO out to larger radii. We are now **explaining these effects by considering the epicyclic frequencies of circular orbits around rotating SQS and we extended our numerical investigation to low-mass models to compare our results with the Newtonian theory of rapidly rotating figures of equilibrium.** The orbital and epicyclic frequencies are of great interest in the context of X-ray observation of high-frequency variability of emission, specifically of the kHz QPOs [13, 18, 21, 40, 42–45].

Our relativistic numerical calculations of the orbital frequencies around low-mass stars, are found to agree well with the analytical calculations for Maclaurin spheroids, given in [24] and [29]. In the Newtonian limit, the gap between the marginally stable orbit and the stellar surface is produced by the oblateness of the rapidly rotating low-mass quark stars [22–24].

VIII. CONCLUSIONS

We have computed numerical models of rapidly rotating strange quark stars and their external metric. In particular, we have computed the orbital and epicyclic frequencies of circular prograde orbits around SQS for astrophysically relevant stellar masses, i.e., those occurring in LMXBs.

We have validated the epicyclic frequency module of the RNS code by comparing the code results for low-mass quark star models, at $M = 0.01M_\odot$ and $M = 0.001M_\odot$, with analytical formulae.

We find that the properties of orbital and epicyclic frequencies are a result of the interplay of competing GR and Newtonian effects. For moderately rotating massive stars the behavior of the epicyclic frequencies is very similar to that of the frequencies in neutron stars, which in

turn are similar to those of prograde orbits in the Kerr metric of slowly spinning black holes ($0 < a_* \ll 1$). However, at high rotation rates the behavior of the epicyclic frequencies near the quark star is similar to that of retrograde orbits in the Kerr metric, i.e., in the latter case the marginally stable orbit is pushed away from the star and the vertical epicyclic frequency is higher than the orbital one.

For moderately rotating massive SQS the behavior of the epicyclic frequencies is dominated by GR effects. However, for rapidly rotating SQS a qualitatively new effect appears for prograde orbits—the vertical epicyclic frequency becomes larger than the orbital frequency. This is a non-relativistic effect of oblateness, known from a study of Maclaurin spheroids [29], as has been verified in a calculation of the frequencies of low mass stars, for which the effects of GR are unimportant, and for which it is found the vertical epicyclic frequency is always larger than the orbital frequency, even at very low stellar rotation rates (Figs. 5–7).

The competition of GR effects and those of higher multipoles can clearly be seen in a plot of the difference between the vertical epicyclic frequency and the orbital one. Fig. 8 shows that for a $1.4M_\odot$ star, at large radii the vertical epicyclic frequency is lower than the orbital one, even at a rotation rate as high as 1165 Hz. It is only close to the star that the effect of higher multipoles prevails and raises the value of the vertical epicyclic frequency above the orbital one. Were this effect, affecting the kHz QPOs (§ VI), to occur also in neutron stars, it could have observable implications in many LMXBs.

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